NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1286

METHOD OF SUCCESSIVE APPROXIMATIONS FOR THE SOLUTION OF CERTAIN PROBLEMS IN AERODYNAMICS

By M. E. Shvets

Translation

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METHOD OF SUCCESSIVE APPROXIMATIONS FOR THE SOLUTION OF CERTAIN PROBLEMS IN AERODYNAMICS*

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The approximate solution of certain problems in the boundary-layer theory is considered herein. The method used is a combination of boundary-layer and successive-approximation methods. The essential character of the method will be evident from the problems considered.

1. Solution of diffusion equation. - A number of physical processes are described by the diffusion equation

$$u' \frac{\partial q'}{\partial x'} = k' \frac{\partial^2 q'}{\partial z'^2} \tag{1.1}$$

and the boundary conditions

$$\left.\begin{aligned} q'\ (0,x) &= Q = \text{constant} \\ q'\ (\infty,x) &= 0 \\ q'\ (z,0) &= 0 \end{aligned}\right\} \tag{1.2}$$

where q is the concentration and u the velocity. Expressed in nondimensional magnitudes

$$q' = Qq$$

$$z' = \frac{k'}{u'} z$$

$$x' = \frac{k'}{u'} x$$

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equation (1.1) assumes the form

$$\frac{\partial^2 q}{\partial z^2} = \frac{\partial q}{\partial x} \tag{1.3}$$

The solution for boundary conditions (equation (1.2)) is

$$q = 1 - \psi\left(\frac{z}{2\sqrt{x}}\right) \qquad \left(\psi(\xi) = \frac{2}{\sqrt{\pi}}\int_0^\xi e^{-\lambda^2}\,d\lambda\right) \tag{1.4}$$

where ψ is the error function.

This solution gives the following expression for the diffusion velocity:

$$\left(\frac{\partial q}{\partial z}\right)_0 = \frac{1}{\sqrt{\pi x}} = \frac{0.57}{\sqrt{x}} \tag{1.5}$$

In considering the approximate solution of this problem, function δ termed the "thickness of the boundary layer" is introduced and the condition for $z = \infty$ is written as the condition for $z = \delta$. The following conditions are thus obtained:

$$\left.\begin{aligned} q\ (0,x) &= 1 \\ q\ (\delta,x) &= 0 \end{aligned}\right\} \tag{1.6}$$

The thickness of the boundary layer δ is an unknown function of the variable x; in order to determine it, the condition

$$\left(\frac{\partial q}{\partial z}\right)_\delta = 0 \tag{1.7}$$

must be assumed.

With the aid of successive approximations, an attempt will be made to obtain q:

$$\left.\begin{aligned} q^{(1)} &= q_0 \\ q^{(2)} &= q_0 + q_1 \\ q^{(3)} &= q_0 + q_1 + q_2 \ldots \\ q^{(n+1)} &= q_0 + \ldots + q_n \end{aligned}\right\} \qquad (1.8)$$

where each of the functions is determined by equations (1.9)

$$\left.\begin{aligned} \frac{\partial^2 q_0}{\partial z^2} &= 0 \\ \frac{\partial^2 q_1}{\partial z^2} &= \dot{q}_0 \\ \frac{\partial^2 q_2}{\partial z^2} &= \dot{q}_1 \ldots \\ \frac{\partial^2 q_{n+1}}{\partial z^2} &= \dot{q}_n \end{aligned}\right\} \qquad (1.9)$$

where the dot denotes differentiation with respect to x and satisfies the boundary conditions of equations (1.6).

The operation of finding q from the given ζ with the aid of equations (1.9) and all boundary conditions of the problem are denoted $\Phi(\zeta)$.

Equations (1.9) then assume the form

$$\left.\begin{aligned} q_0 &= 1 - \frac{z}{\delta} \\ q_1 &= \Phi(\dot{q}_0) \ldots \\ q_{n+1} &= \Phi(\dot{q}_n) \end{aligned}\right\} \qquad (1.10)$$

The process of successive approximation considered gives the series

$$q = q_0 + \Phi(\dot{q}_0) + \Phi(\dot{q}_1) + \ldots + \Phi(\dot{q}_n) \tag{1.11}$$

Stopping at the second approximation gives the following expression for q:

$$q = 1 - \frac{z}{\delta} + \frac{z\dot{\delta}}{6}\left(\frac{z^2}{\delta^2} - 1\right) \tag{1.12}$$

For boundary-layer thickness from the condition of equation (1.7)

$$\left.\begin{aligned} \delta\dot{\delta} &= 3 \\ \delta &= 2.45\sqrt{x} \end{aligned}\right\} \tag{1.13}$$

For the value of δ determined, the following equations are obtained from equation (1.12) for the diffusion velocity and the concentration q:

$$\left.\begin{aligned} -\left(\frac{\partial q}{\partial z}\right)_0 &= \frac{1}{\delta} + \frac{\delta}{6} \approx \frac{0.61}{\sqrt{x}} \\ q &= 1 - \frac{\xi}{2.45} + \frac{2.45}{12}\xi\left[\frac{\xi^2}{6} - 1\right] \\ &\left(\xi = \frac{z}{\sqrt{x}}\right) \end{aligned}\right\} \tag{1.14}$$

By comparing the diffusion velocity computed by the exact equation (1.4) and the approximate equation (1.14) with the corresponding curves in figure 1, the effectiveness of the approximation method can be seen.

<u>2. Approximate integration of boundary-layer equation of flat plate in incompressible flow.</u> - A flat plate moves with constant velocity U. The equations of motion and continuity for the laminar boundary layer will be

$$\left.\begin{aligned} u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} &= \nu \frac{\partial^2 u'}{\partial y'^2} \\ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} &= 0 \end{aligned}\right\} \tag{2.1}$$

where ν is the kinematic viscosity.

Elimination of the vertical component of velocity gives

$$\nu \frac{\partial^2 u'}{\partial y'^2} = u' \frac{\partial u'}{\partial x'} - \frac{\partial u'}{\partial y'} \int_0^{y'} \frac{\partial u'}{\partial x'} dy' \tag{2.2}$$

The nondimensional magnitudes

$$\left.\begin{aligned} x' &= lx & y' &= \frac{l}{\sqrt{R}} y \\ v' &= v \frac{U}{\sqrt{R}} & u' &= Uu \end{aligned}\right\} \tag{2.3}$$

are introducted where l is the length of the plate and R is the Reynolds number. Equation (2.2) assumes the form

$$\frac{\partial^2 u}{\partial y^2} = u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \int_0^{y} \frac{\partial u}{\partial x} dy \tag{2.4}$$

The solution of equation (2.4) must satisfy the conditions

$$\left.\begin{aligned} u &= 0 \quad \text{for} \quad y = 0 \\ u &= 1 \quad \text{for} \quad y = \delta \end{aligned}\right\} \tag{2.5}$$

The first approximation is obtained by solving the equation $\partial^2 u/\partial y^2 = 0$ for the boundary conditions of equations (2.5). This approximation has the form $u = y/\delta$. By substituting the first approximation for u in the right side of equation (2.4) and carrying out two quadratures, the second approximation is obtained:

$$u = \frac{y}{\delta} + \frac{\dot{\delta} y}{24} \left(1 - \frac{y^3}{\delta^3}\right) \tag{2.6}$$

As is shown by computation, the third approximation is not required. The magnitude δ is determined from the condition $(\partial u/\partial y)_\delta = 0$.

$$\delta = 4\sqrt{x} \tag{2.7}$$

The frictional stress τ according to equation (2.6) is

$$\tau = \left(\frac{\partial u}{\partial y}\right)_0 = \frac{1}{\delta} + \frac{\dot{\delta}}{24} = \frac{0.333}{\sqrt{x}} \tag{2.8}$$

or in dimensional magnitudes

$$\tau' = \sqrt{\mu\rho U^3}\,\frac{0.333}{\sqrt{x}} \tag{2.9}$$

This equation differs from the exact numerical coefficient, which according to Blasius (reference 1) is 0.332.

Substituting the value of δ in equation (2.6) for the velocity profile yields

$$u = \frac{1}{3}\xi - \frac{1}{768}\xi^4 \qquad \left(\xi = \frac{y}{\sqrt{x}}\right) \tag{2.10}$$

The vertical component of the velocity is determined from the condition of continuity

$$v = \frac{1}{12\sqrt{x}}\left[\xi^2 - \frac{1}{160}\xi^5\right] \tag{2.11}$$

In figure 2 are shown the velocity profiles obtained by the approximate and exact solutions. The exact solution is mathematically rather complicated.

<u>3. Motion of fluid in laminar boundary layer with longitudinal pressure drop.</u> - In this case, the motion is described by the equation

$$\frac{\partial^2 u}{\partial y^2} = u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\,dy - U\frac{dU}{dx}$$

with the boundary conditions

$$u = 0 \quad \text{for} \quad y = 0$$

$$u = U \quad \text{for} \quad y = \delta$$

If the calculation is restricted to the second approximation,

$$\frac{u}{U} = \frac{\dot{U}}{24}\delta^2\left(\xi^4 - 12\xi^2 + 11\xi\right) - \frac{U\dot{\delta}\delta}{24}\left(\xi^4 - \xi\right) + \xi \qquad (\xi = y/\delta)$$

The equation for determining the boundary-layer thickness, which is obtained from the condition $(\partial u/\partial \xi)_{\xi=1} = 0$ has the form

$$\frac{3}{8}\delta^2\dot{U} + \frac{1}{8}U\dot{\delta}\delta = 1$$

If it is assumed that $\delta = 0$ for $x = 0$

$$\delta^2 = \frac{16}{U^6}\int_0^x U^5 dx$$

For the frictional stress at the wall

$$\tau_0 = \left(\frac{\partial u}{\partial y}\right)_0 = \frac{U}{3}\left[\frac{4}{\delta} + \delta\dot{U}\right] = \frac{U^4}{3}\left(\int_0^x U^5 dx\right)^{-1/2}\left[1 + \frac{4\dot{U}}{U^6}\int_0^x U^5 dx\right]$$

The point of separation of the boundary layer is obtained from the condition

$$\frac{4\dot{U}}{U^6}\int_0^x U^5 dx = -1$$

<u>4. Heat-transfer in laminar boundary layer on plate in incompressible fluid, constants of which are independent of temperature.</u> - If the addition of heat due to friction is neglected and the process is considered steady, the equation of heat conductivity is obtained:

$$u'\frac{\partial T}{\partial x'} + v'\frac{\partial T}{\partial y'} = k\frac{\partial^2 T}{\partial y'^2} \tag{4.1}$$

When the nondimensional magnitudes (equations (2.3)) and the nondimensional temperature

$$\frac{T_1 - T}{T_1 - T_0} = \vartheta \tag{4.2}$$

where T_1 is the absolute temperature of the plate, and T_0 the absolute temperature of the fluid on the edge of the boundary layer, are introduced equation (4.1) assumes the form

$$u \frac{\partial \vartheta}{\partial x} + v \frac{\partial \vartheta}{\partial x} = \frac{1}{P} \frac{\partial^2 \vartheta}{\partial y^2} \tag{4.3}$$

where P is the Prandtl number. The boundary conditions will be

$$\vartheta = 0 \quad \text{for} \quad y = 0$$

$$\vartheta = 1 \quad \text{for} \quad y = \delta$$

If the values of the velocities according to Blasius (reference 1) are substituted in equation (4.3), the equation is solved and leads to the following result:

$$\vartheta(\xi) = C(P) \int_0^{\xi} \exp\left\{\frac{-P}{2} \int_0^{\xi} \varphi \, d\eta\right\} d\xi \tag{4.4}$$

where

$$\left.\begin{aligned} C(P) &= \left[\int_0^{\infty} \exp\left\{\frac{-P}{2} \int_0^{\xi} \varphi \, d\eta\right\} d\xi\right]^{-1} \\ \varphi'(\xi) &= u \\ \xi &= \frac{y}{\sqrt{x}} \end{aligned}\right\} \tag{4.5}$$

As is seen from these relations, for a given Prandtl number P the temperature and the flow of heat can be obtained only by the method of numerical integration. The solution is rendered considerably more complicated if, in equation (4.1), the effect of the heat due to the work of friction is considered.

By the interpolation of Pohlhausen in the interval $0.6 < P < 15.0$, the function $C(P)$ may be represented in the form

$$C(P) = 0.332 \sqrt[3]{P}$$

The problem of the heat-transfer of the plate has been the subject of many investigations, for example references 2 and 3. Piercy and Schmidt (reference 4) applying an approximate method similar to the method of Oseen arrived at the result

$$C(P) = 0.404 \sqrt[3]{P}$$

This result is about 20 percent greater than the result of Pohlhausen.

The problem under consideration will now be solved by the approximate method. As a first approximation, $\vartheta = y/\delta$.

Substituting the values of the component velocities obtained in section 2 and the derivatives of the temperature computed by the first approximation in the left side of equation (4.3) yields the second approximation:

$$\vartheta = \frac{y}{\delta} + \frac{P}{9\delta\sqrt{x}}\left\{\frac{y^4}{4}\left(\frac{1}{4x} - \frac{\dot{\delta}}{\delta}\right) + \frac{y^7}{1792x^{3/2}}\left(\frac{\dot{\delta}}{2\delta} - \frac{1}{5x}\right)\right\} + Cy \tag{4.6}$$

where

$$C = \frac{P}{9\delta\sqrt{x}}\left\{\frac{\delta^3}{\delta}\left(\frac{\dot{\delta}}{\delta} - \frac{1}{4x}\right) + \frac{1}{1792x^{3/2}}\left(\frac{1}{5x} - \frac{\dot{\delta}}{2\delta}\right)\delta^6\right\} \tag{4.7}$$

From the condition $(\partial T/\partial y)_\delta = 0$, boundary-layer thickness is obtained

$$\delta \approx \frac{3.63}{P^{1/3}}\sqrt{x} \tag{4.8}$$

At a Prandtl number $P = 1$, the boundary-layer thickness is 10 percent less than in the dynamic problem of the flow about a plate. This difference is explained by the fact that the solution (4.6) is approximate. If $P = 0.8$ (air) is assumed, the boundary-layer thickness will be

$$\delta = 4\sqrt{x}$$

By differentiating equation (4.6) with respect to y and setting $y = 0$, the following expression for the flow is obtained:

$$q = \left(\frac{\partial T}{\partial y}\right)_0 = 0.368 \sqrt[3]{P} \frac{1}{\sqrt{x}}$$

When this result is compared with the results obtained by others (Piercy and Schmidt, reference 4), it is found to be closer to the accurate value.

Substituting equation (4.8) in equation (4.6) yields the temperature profile

$$\vartheta = \frac{\xi}{\beta} - \frac{P}{144\beta}\left[(\xi^4 - \beta^3\xi) - \frac{1}{2240}(\xi^7 - \beta^6\xi)\right] \quad \left(\beta \approx \sqrt[3]{\frac{48}{P}}\right) \tag{4.9}$$

In figure 3 are given the profiles of the nondimensional temperature ϑ for the values $P = 1$ and $P = 7$. The solid curves are computed by the approximate formula (4.6) and the dashed ones by the accurate formula. These curves lie close to each other.

5. Cooling of heated sphere by laminar flow of fluid at small values of Reynolds number. - The problem of the slow steady laminar flow about a sphere was considered by Stokes (reference 5). The magnitude of the tangential component of the velocity of the fluid at the distance r from the center of a spherical drop of radius r_0 is determined by the expression

$$u_\theta = U \sin\theta \left[1 + \frac{3}{4}\frac{r_0}{r} - \frac{1}{4}\frac{{r_0}^3}{r^3}\right] \tag{5.1}$$

The magnitude y is introduced by the relation $r = r_0 + y$. For small value of y (near the surface of the sphere)

$$u_\theta = 3U \frac{y}{d} \sin\theta \tag{5.2}$$

In accordance with Leibenson (reference 6), the equation of heat conductivity is presented in the form

$$u_\theta \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} \tag{5.4}$$

or, substituting the value of u_0

$$U \frac{3 \sin\theta}{kd} y \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} \tag{5.5}$$

The solution of this equation must satisfy the following boundary and initial conditions:

$$\left.\begin{aligned} T &= T_0 \quad \text{for} \quad y = 0 \\ T &= 0 \quad \text{for} \quad y = \infty \\ T &= 0 \quad \text{for} \quad x = 0 \end{aligned}\right\} \tag{5.6}$$

For simplification of the integration of equation (5.5), in accordance with Leibenson (reference 6), the magnitude $(3/kd) \sin \theta$ is at first assumed constant. The following variables are introduced:

$$\left.\begin{aligned} y &= 2\sqrt{\tau} \\ \frac{kd\ x}{U\ 6 \sin \theta} &= \xi \end{aligned}\right\} \tag{5.7}$$

Instead of equation (5.5) equation (5.8) is obtained:

$$\frac{\partial T}{\partial \xi} = \frac{\partial}{\partial \tau} \sqrt{\tau}\, \frac{\partial T}{\partial \tau} \tag{5.8}$$

The solution of this equation is now obtained by the approximate method. As previously, the boundary-layer thickness is denoted by δ. If the calculation is restricted to the second approximation, the approximate solution of equation (5.8) is obtained:

$$T = 1 - \sqrt{\frac{\tau}{\delta}} + \frac{\dot{\delta}\sqrt{\tau}}{6}\left[\left(\frac{\tau}{\delta}\right)^{3/2} - 1\right] \tag{5.9}$$

Determining the boundary-layer thickness from equating the flow at the edge of the boundary layer to zero yields the equation

$$\left.\begin{aligned} \delta^{1/2}\, \dot{\delta} &= 2 \\ \delta &= 2.08\, \xi^{2/3} \end{aligned}\right\} \tag{5.10}$$

When the value of $\dot{\delta}$ is substituted in equation (5.9)

$$T = 1 - \frac{4}{3}\left(\frac{\tau}{\delta}\right)^{1/2} + \frac{1}{3}\left(\frac{\tau}{\delta}\right)^{2} \tag{5.11}$$

Computing the velocity of the flow of heat and returning to the previous variables yields in succession

$$-\left(\sqrt{\tau}\ \frac{\partial T}{\partial \tau}\right)_0 = \frac{2}{3\delta^{1/2}} = \frac{0.46}{\xi^{1/3}} \tag{5.12}$$

or

$$-\left(\sqrt{\tau}\ \frac{\partial T}{\partial \tau}\right)_0 = 0.84 \left(\frac{U \sin \theta}{x\ kd}\right)^{1/3}$$

The heat given off by the sphere for small values of the Reynolds number R for air was considered by Leibenson. As a result of the accurate solution of equation (5.8), he obtained (reference 6, p. 280) equation (5.12) but with the numerical coefficient 0.85.

6. Free convection at plane vertical wall. - Let the origin of coordinates be at the edge of the plate; the x-axis is directed along the plane of the plate; the y-axis is perpendicular to the x-axis.

From the physical considerations it is clear that the velocity and the temperature difference $T - T_0$ (where T_0 is the temperature of the air at large distances from the plate) differ appreciably from zero only in the thin boundary layer at the surface of the plate.

For this reason, the process of free convection is described by the boundary-layer equations

$$\left.\begin{aligned} u' \frac{\partial u'}{\partial x'} + v \frac{\partial u'}{\partial y'} &= \nu \frac{\partial^2 u'}{\partial y'^2} + g\beta\,(T' - T_0) \\ u' \frac{\partial T'}{\partial x'} + v' \frac{\partial T'}{\partial y'} &= k \frac{\partial^2 T'}{\partial y'^2} \\ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} &= 0 \end{aligned}\right\} \tag{6.1}$$

For the boundary conditions

$$\left.\begin{aligned} u = 0 \quad v = 0 \quad T = T_1 \quad &\text{for} \quad y = 0 \\ u = 0 \quad T = T_0 \quad &\text{for} \quad y = \infty \end{aligned}\right\} \tag{6.2}$$

where T_1 is the temperature of the plate.

From the parameters entering the equations, measures of length and velocity can be constructed:

$$\left.\begin{aligned} &\left[\frac{\nu^2}{g\beta\,(T_1 - T_0)}\right]^{1/3} \\ &\left[\nu g(T_1 - T_0)\beta\right]^{1/3} \end{aligned}\right\} \tag{6.3}$$

Equations (6.1) reduce to the form

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\,dy &= \frac{\partial^2 u}{\partial y^2} + \vartheta \\ u\frac{\partial \vartheta}{\partial x} - \frac{\partial \vartheta}{\partial y}\int_0^y \frac{\partial u}{\partial x}\,dy &= \frac{1}{P}\frac{\partial^2 \vartheta}{\partial y^2} \end{aligned}\right\} \tag{6.4}$$

where

$$\left.\begin{aligned} \vartheta &= \frac{T' - T_0}{T_1 - T_0} \\ \vartheta\,(0) &= 1 \\ \vartheta\,(\infty) &= 0 \end{aligned}\right\} \tag{6.5}$$

The boundary layer of convection δ is next considered and the problem is solved by the approximate method.

The first approximation is

$$\left.\begin{aligned} \vartheta &= 1 - \frac{y}{\delta} \\ u &= \frac{y^3}{6\delta} - \frac{y^2}{2} + \frac{\delta y}{3} \end{aligned}\right\} \tag{6.6}$$

Substituting the values of u and ϑ in the left side of the second equation of equations (6.4) yields the second approximation

$$\vartheta = 1 - \frac{y}{\delta} + P\left\{\frac{\dot{\delta}}{\delta^2}\left(\frac{y^6}{240\delta} - \frac{y^5}{40} + \frac{\delta y^4}{24}\right) - \frac{\dot{\delta}\delta^2}{48}\, y\right\} \tag{6.7}$$

The determination of the second approximation for the velocity, which is obtained without difficulty, is omitted. The boundary-layer thickness is determined from the condition $(\partial\vartheta/\partial y)_\delta = 0$. Thus

$$\delta^3\dot{\delta} = \frac{240}{11P}$$

or (6.8)

$$\delta = 3.06\left(\frac{x}{P}\right)^{1/4}$$

Substituting the value of $\dot{\delta}$ in equation (6.7) yields the temperature profile

$$\vartheta = 1 - \frac{16}{11}\left(\frac{y}{\delta}\right) + \frac{1}{11}\left(\frac{y}{\delta}\right)^6 - \frac{6}{11}\left(\frac{y}{\delta}\right)^5 + \frac{10}{11}\left(\frac{y}{\delta}\right)^4 \tag{6.9}$$

whence the rate of heat flow will be

$$-\left(\frac{\partial\vartheta}{\partial y}\right)_{y=0} = 0.48\left(\frac{x}{P}\right)^{1/4}$$

The accurate solution of the problem of the heat-transfer from a heated vertical wall was obtained by Pohlhausen (reference 7), who showed that if the magnitude

$$\xi = C\frac{y}{x^{1/4}} \qquad \left(C = \left[\frac{\beta g(T_1 - T_0)}{4\nu^2}\right]^{1/4}\right) \tag{6.10}$$

is introduced as an independent variable and in place of the required functions, new functions defined by the relations

$$u = 4\nu C^2\sqrt{x}\,\varphi'(\xi)$$

$$\frac{T - T_0}{T_1 - T_0} = \vartheta(\xi)$$

are substituted, the initial system of equations takes the form

$$\left.\begin{aligned} \varphi''' + 3\varphi\varphi'' - 2\varphi'^2 + \vartheta &= 0 \\ \vartheta'' + 3P\varphi\vartheta' &= 0 \end{aligned}\right\} \tag{6.11}$$

The solution of this system is obtained by numerical integration. From equations (6.10) and (6.11), the boundary-layer thickness will be of the order $\delta \sim x^{1/4}$. In figure 4, taken from reference 8, is given the curve of ϑ for air according to Pohlhausen. On the same figure the dashed curve shows the graph of ϑ computed by equation (6.9); for $y/\delta < 3/4$ both curves practically coincide.

7. Stationary turbulent diffusion. - In the solution of certain problems of geophysics, it is necessary to deal with the equation of diffusion in which the so-called coefficient of turbulent exchange A enters instead of a constant coefficient of diffusion. For steady conditions, A depends only on the distance z.

The problem is restricted to obtaining the second approximation, which is entirely sufficient for practical purposes. The problem of the character of the dependence of the coefficient of turbulent exchange on the distance z is not considered. Both sides of the diffusion equation

$$u(z)\frac{\partial q'}{\partial x} = \frac{\partial}{\partial z}A(z)\frac{\partial q'}{\partial z} \tag{7.1}$$

are multiplied by $A(z)$ and the new variable is introduced:

$$\eta = \int_0^z \frac{dz}{A} \tag{7.2}$$

Equation (7.1) then assumes the form

$$\left.\begin{aligned} \Phi(\eta)\frac{\partial q'}{\partial x} &= \frac{\partial^2 q'}{\partial \eta^2} \\ (uA &= \Phi(\eta)) \end{aligned}\right\} \tag{7.3}$$

Let the boundary and initial conditions be

$$\left.\begin{aligned} q' &= q_0 \quad \text{for} \quad z = 0 \\ q' &= 0 \quad \text{for} \quad z = \infty \\ q' &= 0 \quad \text{for} \quad x = 0 \end{aligned}\right\} \tag{7.4}$$

In order to simplify these conditions, $q' = q_0 q$; then

$$q = 1 \quad \text{for} \quad z = 0$$

At the edge of the boundary layer, the condition must exist that $q = 0$, that is, that $q\ (\delta) = 0$.

The first approximation has the form

$$q = 1 - \frac{\eta}{\delta} \tag{7.5}$$

The second approximation is given by

$$q = 1 - \frac{\eta}{\delta} + \frac{\dot{\delta}}{\delta}\left[\int_0^\eta y(\eta - y)\Phi\, dy - \frac{\eta}{\delta}\int_0^\delta y(\delta - y)\,\Phi\, dy\right] \tag{7.6}$$

$$\frac{\partial q}{\partial \eta} = -\frac{1}{\delta} + \frac{\dot{\delta}}{\delta^2}\left[\int_0^\eta y\,\Phi\, dy - \frac{1}{\delta}\int_0^\delta y(\delta - y)\Phi\, dy\right] \tag{7.7}$$

The condition $(\partial q/\partial \eta)_\delta = 0$ gives for the boundary-layer thickness δ the equation

$$\dot{\delta}\int_0^\delta y^2\,\Phi dy = \delta^2 \tag{7.8}$$

From equation (7.7) for $\eta = 0$,

$$-\left(\frac{\partial q}{\partial \eta}\right)_0 = \frac{1}{\delta} + \frac{\dot{\delta}}{\delta^3}\int_0^\delta y(\delta - y)\Phi\, dy \tag{7.9}$$

By making use of equation (7.8), equations (7.9) and (7.6) for the concentration may be transformed into the form

$$
\left.\begin{aligned}
-\left(\frac{\partial q}{\partial \eta}\right)_0 &= \frac{1}{\delta} + \frac{\dot{\delta}}{\delta^2}\int_0^\delta y\,\Phi\,dy = \frac{J_1}{J_2} \\
q &= 1 + \frac{1}{J_2}\int_0^\eta y(\eta - y)\,\Phi\,dy - \eta\,\frac{J_1}{J_2}
\end{aligned}\right\} \tag{7.10}
$$

where

$$J_1 = \int_0^\delta y\,\Phi\,dy$$

$$J_2 = \int_0^\delta y^2\,\Phi\,dy$$

Comparison with the exact solutions shows the effectiveness of the given approximation method.

Translated by S. Reiss,
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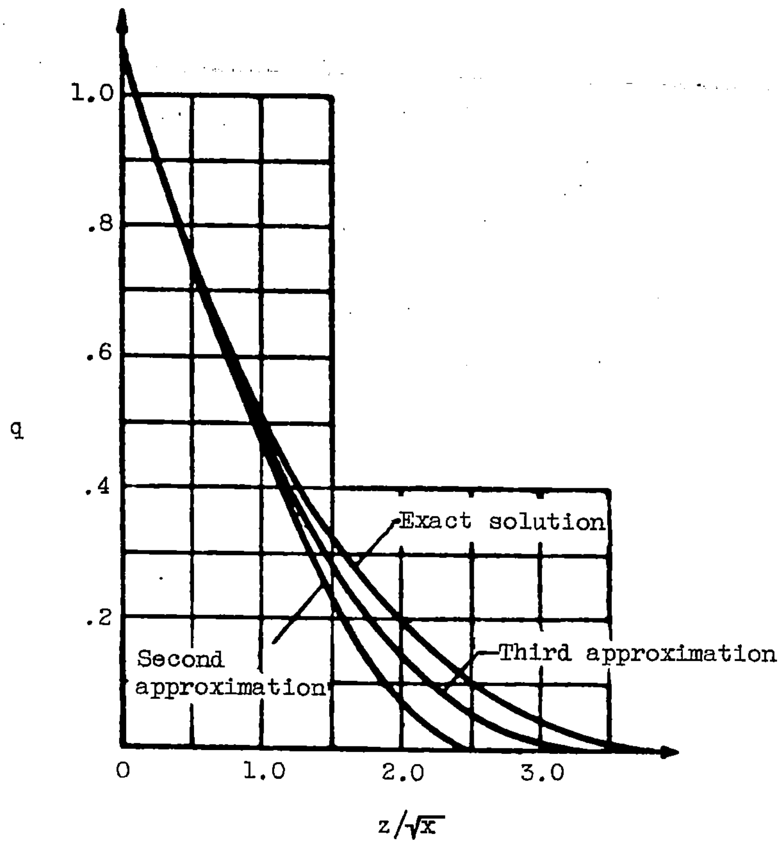

Figure 1.

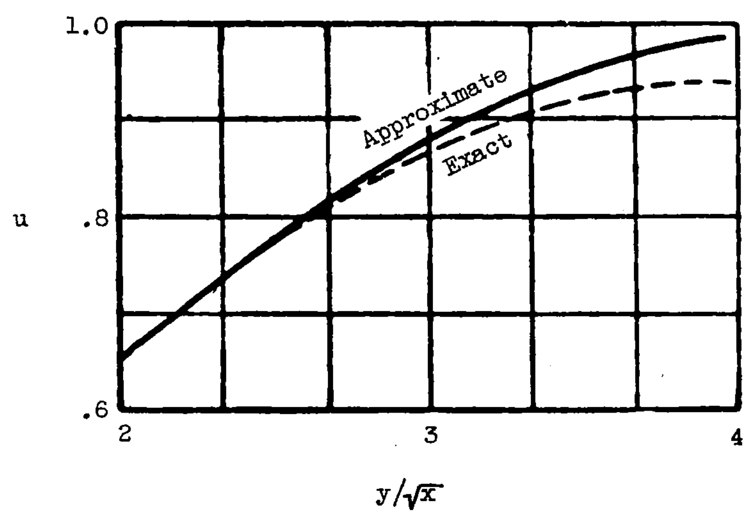

Figure 2.

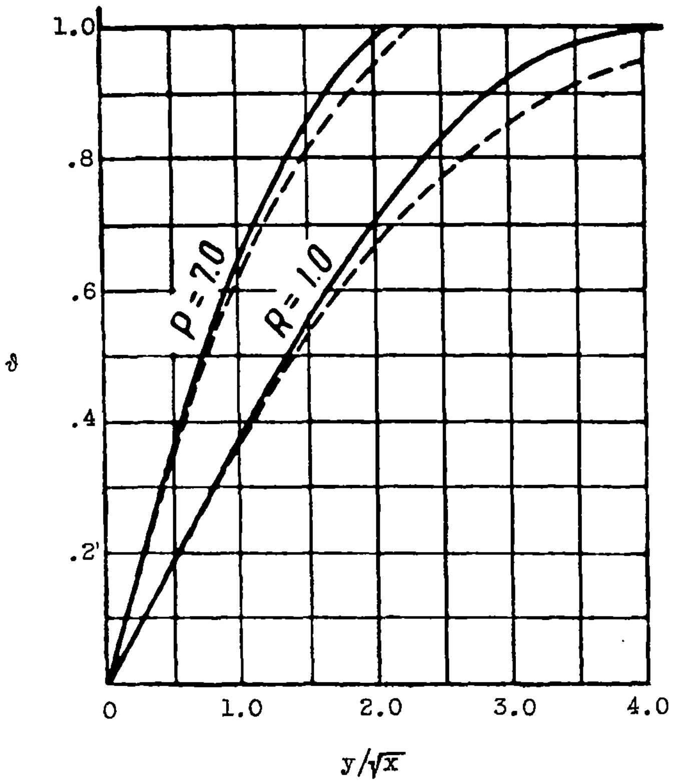

Figure 3.

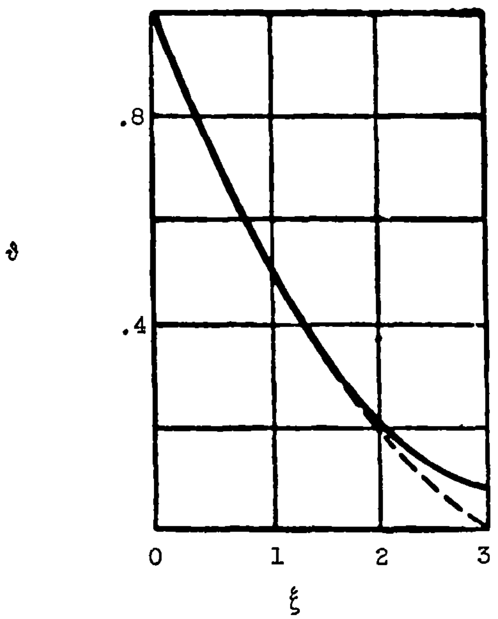

Figure 4.